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THE BOUNDARY CONDITION AT AN IMPEDANCE WALL IN A NON-UNIFORM DUCT WITH POTENTIAL MEAN FLOW

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The boundary condition at an impedance wall in a duct with a steady mean flow requiring the specification of the normal component of acoustic particle velocity is examined. It is found that when implemented in the weak formulation of the finite element method it can be considerably simplified. The boundary condition would appear to require data which includes the tangential derivative of the tangential mean flow velocity, the normal derivative of the normal component of mean flow velocity, and the derivatives of the mean flow density and the boundary admittance along the boundary. It is shown that with suitable rearrangement the normal and tangential velocity derivatives can be eliminated, as can the derivatives of the mean flow density and admittance. The boundary condition becomes only slightly more complicated than the corresponding boundary condition when mean flow is absent, and is no more difficult to implement, requiring only local values of tangential mean flow velocity, density, and admittance which are already required as data for the weak formulation of the field equation.

1. INTRODUCTION

Figure 1 shows the geometry of a typical non-uniform duct section. The duct is of non-uniform cross section with walls S_w which in general include an acoustically absorbing section imbedded in an otherwise acoustically rigid wall. Absorption characteristics of the boundary are given in terms of admittance A for a locally reacting liner. The duct in Figure 1 is depicted as axially symmetric; however, the results obtained here do not depend on such an idealization. The duct geometry includes the definition of a unit normal **n** directed out of the fluid region, and therefore into the duct wall. The notional displacement of the duct wall, normal to the wall, is given by ζ which is a function of location on the wall. In harmonic motion with time dependence $e^{i\eta_r t}$, the admittance relates this displacement to the acoustic pressure according to

$$i\eta_r \zeta = Ap. \tag{1}$$

When this admittance relation is applied to acoustic propagation in ducts with steady mean flow, it produces what appears to be a very difficult boundary condition at the admittance wall. Myers [1] derived the correct boundary condition which relates the normal component of acoustic particle velocity to the particle displacement in non-viscous flow for harmonic acoustic perturbations at frequency η_r as

$$\mathbf{v}\cdot\mathbf{n} = i\eta_r \zeta + \mathbf{V}_r \cdot \nabla\zeta - \zeta\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{V}_r. \tag{2}$$

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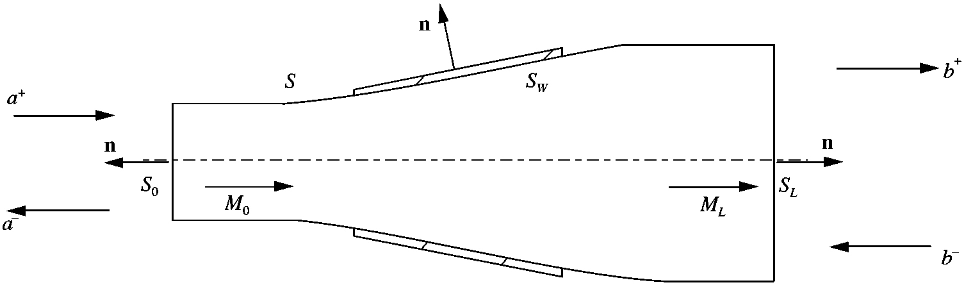

Figure 1. An x, r section of a non-uniform duct showing essential modelling features, including acoustic treatment and steady mean flow.

Here, $\mathbf{v} \cdot \mathbf{n}$ is the normal component of acoustic particle velocity at the wall and $\mathbf{V}_r$ is the mean flow velocity, tangential at the wall. Propagation in non-uniform ducts is normally modelled under the assumption that the mean flow and acoustic perturbation are defined by a steady flow potential such that $\mathbf{V}_r = \nabla\phi_r$, and by an acoustic potential such that $\mathbf{v} = \nabla\phi$ and an acoustic momentum equation

$$p = -\rho_r[\mathrm{i}\eta_r\phi + \mathbf{V}_r \cdot \nabla\phi]. \tag{3}$$

A combination of equations (1)–(3) produces a single boundary condition in terms of acoustic potential which is difficult to implement in numerical schemes. It is found that the data required to model the boundary condition include the derivative of the impedance and mean flow density along the boundary. In addition, and much more of a problem, is the requirement for the tangential derivative of the tangential component of mean flow and the normal derivative of the normal component of mean flow. These present particular difficulties because the mean flow data in finite element propagation models are generally obtained from a potential formulation for the steady flow and the essential derivatives of velocity require second derivatives of the potential.

The boundary condition described by equations (1)–(3) has been implemented in a finite element scheme by Eversman and Okunbor [2]. They used an approximation, known to be adequate for ducts with changes in cross-section which are relatively small, which ignores the term requiring the normal derivative of the normal component of mean flow velocity, and additionally ignores the effect of duct wall curvature on the calculation of the rates of change of quantities along the wall. The approximation for the tangential derivative of the tangential component of mean flow velocity is retained. The computation of this derivative is not considered to be very accurate. None of these approximations are thought to be significant for attenuation calculations in the geometries considered.

Rienstra [3] has approached the modelling of acoustic propagation in ducts by slowly varying the cross-section by a perturbation scheme, and the analysis procedure requires the full modelling of the boundary condition. However, because his procedure is analytic the implementation of the boundary condition presents no difficulty and the issues which arise in a numerical model are not present.

The motivation for the present investigation is the requirement to verify a reciprocity relationship which exists for acoustic propagation in non-uniform ducts with mean flow and absorbing linings. In order to show reciprocity, no approximation in the boundary condition is permissible. Numerical experiments conducted with the approximate model of the boundary condition described in reference [2] suggest that reciprocity is nearly satisfied, but one is not fully convinced whether the small discrepancies are in the approximate model or in the reciprocity principle. The following work derives a model of the boundary condition for the FEM formulation which is exact within the FEM formalism, is easy to implement, and will replace the approximation in reference [2]. Work that is to be subsequently reported will show that the new boundary condition results in a numerical substantiation of the reciprocity principle [4, 5].

2. FINITE ELEMENT FORMULATION FOR DUCT PROPAGATION

Application of finite element modelling to acoustic propagation in non-uniform ducts with a steady mean potential flow has been previously reported [2]. A formulation in terms of acoustic potential is used to reduce the field equations to a single scalar variable. In this investigation, the geometry of the duct and steady flow field is axially symmetric. The acoustic field is not axially symmetric but is represented as azimuthally periodic in a cylindrical co-ordinate system with x being the axis of symmetry, r the cylindrical radius in a circular cross-section at $x = 0$, and θ the angular co-ordinate. Solutions are sought in angular harmonics of a Fourier series in θ enumerated by the angular mode number m. This reduces the solution domain to a two-dimensional x, r plane, as shown in Figure 1. The duct shape in a $\theta =$ constant plane shows the surface S which defines the duct shape and could also include an inner surface for an annular duct. Part of S includes S_w, which is a locally reacting acoustic treatment.

The acoustic field is assumed to be harmonic in time at non-dimensional frequency η_r. Geometry is non-dimensional based on a reference length generally chosen as the radius of the inlet at the source plane, R. Acoustic and steady flow variables are non-dimensional based on reference values of the speed of sound and density of the medium, ρ_∞, c_∞, generally defined at the plane of the acoustic source. The non-dimensional frequency is $\eta_r = \omega R/c_\infty$, with ω as the harmonic source frequency.

Reference [2] discusses in detail the finite element modelling of acoustic propagation in and near ducts carrying mean flow. The field equations for continuity and momentum and the isentropic equation of state are used in a weighted residual statement to obtain an integral formulation which is then written in discrete form using standard FEM procedures. In terms of acoustic potential, the weak formulation is

$$\iiint_V \frac{\rho_r}{c_r^2}\{c_r^2 \nabla W \cdot \nabla\phi - (\mathbf{V}_r \cdot \nabla W)(\mathbf{V}_r \cdot \nabla\phi) + \mathrm{i}\eta_r[W(\mathbf{V}_r \cdot \nabla\phi) - (\mathbf{V}_r \cdot \nabla W)\phi] - \eta_r^2 W\phi\}\,\mathrm{d}V$$

$$= \iint_S \frac{\rho_r}{c_r^2}\{c_r^2 W\,\nabla\phi - \mathbf{V}_r W(\mathbf{V}_r \cdot \nabla\phi) - \mathrm{i}\eta_r \mathbf{V}_r W\phi\}\cdot\mathbf{n}\,\mathrm{d}S, \tag{4}$$

where the local non-dimensional steady flow velocity is $\mathbf{V}_r = \nabla\phi_r$, with ϕ_r as the non-dimensional steady flow velocity potential. The local non-dimensional density and the speed of sound are ρ_r, c_r. The surface integral on the right-hand side introduces the noise source and termination conditions on S_0 or S_L and a possible impedance boundary condition on S inside the duct. In the present investigation, it is the impedance boundary condition on S_w, a portion of S which is of interest. In equation (4), the weighted residuals statement, W represents an arbitrary weighting function selected from the class of continuous functions. In this weak formulation, the approximation to the solution ϕ is also chosen from the class of continuous functions.

At a duct wall, the mean flow is tangential to the wall and $\mathbf{V}_r \cdot \mathbf{n} = 0$, causes the boundary integral (the contribution to the right-hand side of equation (1) related to the impedance condition) to become

$$I_b = \iint_{S_w} \rho_r W \nabla\phi \cdot \mathbf{n} \, \mathrm{d}S. \tag{5}$$

With the Myers boundary condition [1] and with $\nabla\phi_r \cdot \mathbf{n} = 0$ on the duct wall surface S_w, the integral of equation (5) on S_w becomes

$$I_b = \iint_{S_w} \{\rho_r W [\mathrm{i}\eta_r \zeta + \mathbf{V}_r \cdot \nabla\zeta - \zeta \mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{V}_r]\} \, \mathrm{d}S. \tag{6}$$

The following vector relations (suggested in a similar context by Moehring [6]) are introduced (with account taken of the special circumstances of the present problem):

$$\rho_r W \mathbf{V}_r \cdot \nabla\zeta = \rho_r \mathbf{V}_r \cdot \nabla W\zeta - \rho_r \zeta \mathbf{V}_r \cdot \nabla W, \quad \nabla \cdot \rho_r \mathbf{V}_r = 0, \qquad \rho_r \mathbf{V}_r \cdot \nabla W\zeta = \nabla \cdot \rho_r W \zeta \mathbf{V}_r,$$

$$\mathbf{n} \cdot \mathbf{V}_r = 0, \tag{7}$$

$$\rho_r W \zeta \mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\mathbf{V}_r = \mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\rho_r W \zeta \mathbf{V}_r,$$

$$\mathbf{n} \cdot \nabla \times (\mathbf{n} \times \rho_r W \zeta \mathbf{V}_r) = \nabla \cdot \rho_r W \zeta \mathbf{V}_r - \mathbf{n} \cdot (\mathbf{n} \cdot \nabla)\rho_r W \zeta \mathbf{V}_r.$$

With the use of the identities of equations (7), equation (6) can then be reformulated as

$$I_b = \iint_{S_w} \{\rho_r \zeta [\mathrm{i}\eta_r W - \mathbf{V}_r \cdot \nabla W]\} \, \mathrm{d}S + \iint_{S_w} \{\mathbf{n} \cdot \nabla \times (\mathbf{n} \times W \rho_r \zeta \mathbf{V}_r)\} \, \mathrm{d}S. \tag{8}$$

Following the development of Moehring [6], the last integral can be written as a line integral on the boundary Γ of the surface S_w by using Stokes' theorem. The boundary curve Γ should enclose the portion of S_w on which there is a non-zero admittance, but should be located where the admittance vanishes, as shown in Figure 2. Γ consists of closed curves Γ_1 and Γ_2 circumscribed on the duct wall at either end of the duct, which is chosen to be outside the region in which the lining of finite length has non-zero admittance, that is, in the regions in which the duct wall is rigid. There is of course a portion of Γ which runs along the duct wall between Γ_1 and Γ_2 to complete the closed curve of Stokes' theorem, but this curve is traversed twice, once in each direction, and has no net contribution. To make use of Stokes' theorem it is required that the acoustic field and the wall displacement be continuous on S_w. Hence, if the acoustic treatment is of limited length imbedded in an otherwise rigid wall duct, the transition from rigid wall to admittance wall, as well as the variation of admittance along the treated wall, must be continuous. If this condition is met, Stokes' theorem can be cast in the form

$$\iint_{S_w} \{\mathbf{n} \cdot \nabla \times (\mathbf{n} \times \rho_r W \zeta \mathbf{V}_r)\} \, \mathrm{d}S = \int_{\Gamma_1} (\mathbf{n} \times \rho_r W \zeta \mathbf{V}_r) \cdot \mathrm{d}\mathbf{\Gamma} + \int_{\Gamma_2} (\mathbf{n} \times \rho_r W \zeta \mathbf{V}_r) \cdot \mathrm{d}\mathbf{\Gamma}. \tag{9}$$

The integral on the surface S_w vanishes if the line integrals vanish. On a hard wall the line integrals vanish because the boundary displacement vanishes. This means that if the

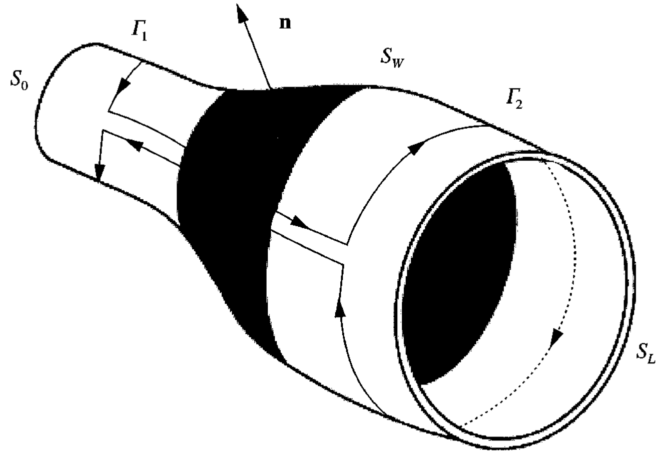

Figure 2. A non-uniform acoustically treated duct segment, showing the integration contour used in the application of Stokes theorem.

condition for the use of Stokes' theorem is met, then the integral of equation (8) is

$$I_b = \iint_{S_w} \{\rho_r \zeta [\mathrm{i}\eta_r W - \mathbf{V}_r \cdot \nabla W]\} \,\mathrm{d}S. \tag{10}$$

At a wall of admittance A equations (1) and (2) are used to replace the wall displacement ζ and the pressure with velocity potential ϕ. The result is the new weighted residual boundary integral on the duct surface S_w,

$$I_b = -\iint_{S_w} A\rho_r^2 \left\{ \mathrm{i}\eta_r W\phi + W\mathbf{V}_r \cdot \nabla\phi - \phi\mathbf{V}_r \cdot \nabla W - \frac{1}{\mathrm{i}\eta_r}(\mathbf{V}_r \cdot \nabla W)(\mathbf{V}_r \cdot \nabla\phi) \right\} \mathrm{d}S. \tag{11}$$

The weighted residual form of the boundary condition of equation (11) is a considerable simplification of the boundary condition which would result by a direct use of equations (1)–(3). In the latter case, it would be found that the derivatives of admittance A and the steady flow density ρ_r are required. In addition, the tangential derivative of the normal component of mean flow velocity tangential to the wall and the normal derivative of the component of mean flow velocity at the wall are required. Current implementations of the FEM formulation from which the steady potential flow field is obtained are not well suited for the accurate determination of these second derivatives of velocity potential. The modified version of the boundary condition neither requires data which is not directly determined from the potential flow model nor requires any operations which are not required in the discretization of the field equations (left-hand side of equation (4)).

The restriction that the admittance is continuous on the duct wall is related to modelling difficulties addressed by other authors. Moehring [6] has noted that in the acoustic potential formulation for discontinuous admittance variation, there is no clear condition to be imposed on the acoustic field or wall displacement at the discontinuity. Rebel and Ronneberger [7] have shown that the condition of admittance discontinuity and the assumption of potential flow at the wall (no boundary layer) cause a problem with the underlying physics of the flow related to the absence of shear stresses. In this analysis, these issues have been eliminated by a prerequisite that the admittance vary continuously. In practical terms, this is accomplished by making "discontinuities" rapid, but continuous, variations (easily done by an appropriate definition of the local admittance). One suspects that numerically this may be a non-issue, because in the weak FEM formulation the role of discontinuities is reduced.

3. AN ALTERNATE APPROACH

An alternate approach to the simplified boundary condition is available which produces a boundary condition useful for numerical models which are not based on the weighted residuals formulation. The Myers boundary condition of equation (4) can be written as

$$\rho_r \mathbf{v}\cdot\mathbf{n} = \mathrm{i}\eta_r\rho_r\zeta + \rho_r\mathbf{V}_r\cdot\nabla\zeta - \rho_r\zeta\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{V}_r. \tag{12}$$

The steady flow continuity equation

$$\nabla\cdot\rho_r\mathbf{V}_r = 0 \tag{13}$$

is used to establish that

$$\rho_r\mathbf{V}_r\cdot\nabla\zeta = \nabla\cdot\rho_r\zeta\mathbf{V}_r. \tag{14}$$

It can also be shown that since on the duct wall $\mathbf{n}\cdot\mathbf{V}_r = 0$,

$$\rho_r\zeta\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{V}_r = \mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\rho_r\zeta\mathbf{V}_r. \tag{15}$$

With these results it can be shown that

$$\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\rho_r\zeta\mathbf{V}_r = \frac{\partial}{\partial n}(\rho_r\zeta V_{r_n}) \tag{16}$$

and

$$\nabla\cdot\rho_r\zeta\mathbf{V}_r = \frac{\partial}{\partial\tau}(\rho_r\zeta V_{r_\tau}) + \frac{\partial}{\partial n}(\rho_r\zeta V_{r_n}). \tag{17}$$

Directions tangential and normal to the duct wall at the wall surface are denoted by τ, n. V_{r_τ}, V_{r_n} are the tangential and normal components of the steady flow velocity. At the duct wall V_{r_n} vanishes. Therefore, the boundary condition on the duct wall is

$$\rho_r\mathbf{v}\cdot\mathbf{n} = \mathrm{i}\eta_r\rho_r\zeta + \frac{\partial}{\partial\tau}(\rho_r\zeta V_{r_\tau}). \tag{18}$$

This form of the boundary condition, not in weighted residual form, could be used, for example, in a finite difference formulation. With ζ replaced by equation (1) and p replaced by equation (3), it is found that derivatives along the wall of mean flow density, wall admittance, and mean flow velocity are required; however, the normal derivative of the normal flow velocity component is not required. Equation (18) can be used in the weighted residual formulation to reproduce equation (11), with the same restrictions.

4. CONCLUSION

The Myers acoustic boundary condition at an admittance wall in a non-uniform duct carrying potential mean flow [1] has been restructured using identities of vector calculus to obtain a form well suited for finite element predictions of propagation. If applied without simplification, the boundary condition would require data on the spatial derivative along the wall of mean flow density, the tangential spatial derivative of the tangential mean flow velocity at the wall, the normal spatial derivative of the normal mean flow velocity at the wall, and the spatial derivative along the wall of the admittance. After simplification, only local values of density, tangential flow velocity and admittance are required. The normal component of mean flow velocity is eliminated completely. Implementation of the boundary condition is easily accomplished in finite element models.

An alternate approach has been used to simplify the Myers boundary condition in a form useful for numerical modelling which was not based on the weighted residuals approach of finite element analysis. The normal derivative of the normal mean flow velocity component at the wall is eliminated; however, derivatives along the wall of mean flow density and velocity and wall admittance are retained.

The net effect of the boundary condition on the prediction of attenuation in ducts in FEM models has been found to be minor when compared to a former approximation introduced for computational efficiency (the new exact formulation is found to be even more computationally simple). For calculations made to validate acoustic reciprocity, the exact form of the boundary condition introduced here is essential, and it is found that the predicted reciprocity relationships are accurately verified [4, 5].

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